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AUTOMORPHIC GREEN FUNCTIONS ON ARITHMETIC QUOTIENTS OF TYPE IV SYMMETRIC DOMAIN

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1. INTRODUCTION

This article is a short summary of the forthcoming paper:

‘Automorphic Green functions associated with the secondary spherical functions’
(Takayuki Oda and Masao Tsuzuki)

Let $G := O_0(n, 2)$ be the identity component of the orthogonal group with signature $(n+, 2-)$ and $K := G \cap \text{diag}(O(n), O(2))$ a maximal compact subgroup of G . The Lie algebra $\mathfrak{g} := \text{Lie}(G)$ is identified with the space of matrices $X \in \text{Mat}_{n+2}(\mathbb{R})$ satisfying ${}^tXI_{n,2} + I_{n,2}X = O$ with the bracket product $[X, Y] = XY - YX$. Let E_{ij} ($1 \leq i, j \leq n+2$) be the usual matrix unit of $\text{Mat}_{n+2}(\mathbb{R})$.

The homogenous manifold G/K is a symmetric space of type IV, which is a Hermitian symmetric domain with the G -invariant complex structure coming from the adjoint action $J := \text{ad}(\tilde{Z}_0)|_{\mathfrak{p}}$ with $\tilde{Z}_0 := E_{n+1,n+2} - E_{n+2,n+1} \in \mathfrak{k} := \text{Lie}(K)$ on \mathfrak{p} , the orthogonal complement of \mathfrak{k} in \mathfrak{g} with respect to the Killing form B of \mathfrak{g} . The K -invariant alternating form $\tilde{\omega}(X, Y) := (8n)^{-1}B(X, J(Y))$ on \mathfrak{p} is uniquely extended to a G -invariant C^∞ differential form ω of $(1, 1)$ type on G/K , by which G/K is a Kähler manifold.

Any arithmetic subgroup Γ of G acts discontinuously on G/K through bi-holomorphic automorphisms of G/K . When Γ is neat, taking the quotient by Γ we have a Kähler manifold $\Gamma \backslash G/K$ with Kähler form $\omega_{\Gamma \backslash G/K}$ such that the quotient map $\pi : G/K \rightarrow \Gamma \backslash G/K$ is holomorphic and $\pi^*\omega_{\Gamma \backslash G/K} = \omega$.

Consider the symmetric subgroup $H = O_0(n-1, 2)$ consisting of fixed points of the involution σ of G defined by $\sigma(g) = SgS$ with $S := \text{diag}(E_{n-1}, -1, E_2)$. We assume that H is ‘ Γ -rational’ in a proper sense. In particular the invariant volume of $\Gamma_H \backslash H/K_H$ is finite, where $\Gamma_H := \Gamma \cap H$ and $K_H := H \cap K$. Let D be the image of the natural holomorphic map $\Gamma_H \backslash H/K_H \rightarrow \Gamma \backslash G/K$. Then D is a closed complex analytic subset of $\Gamma \backslash G/K$ with complex codimension 1, which defines a closed current δ_D by integration

$$\langle \delta_D, \alpha \rangle = \int_{D_{\text{ns}}} j^* \alpha, \quad \alpha \in A_c(\Gamma \backslash G/K).$$

Here D_{ns} denotes the smooth locus of D and $A_c(M)$ denotes the space of compactly supported smooth forms on a complex manifold M .

Then our aim here is to explain an explicit construction of the Green current for D following [1]. Though a similar construction for the ‘unitary case’ (i.e., for the modular divisors in an arithmetic quotient of a complex-hyperball) is proved to be possible, we focus only on the ‘orthogonal case’ setting aside the ‘unitary case’ for simplicity of presentation.

2. SECONDARY SPHERICAL FUNCTIONS

Let \mathfrak{a} be the maximal abelian subspace $\mathbb{R}Y_0$ of $\mathfrak{p} \cap \mathfrak{q}$ with the basis $Y_0 = E_{n,n+1} + E_{n+1,n}$. Here \mathfrak{q} is the orthogonal complement of $\mathfrak{h} := \text{Lie}(H)$. Then the group G is a union of the double cosets Ha_tK ($t \geq 0$) with $a_t := \exp(tY_0)$. We introduce two functions $\phi_s^{(2)}$ and ψ_s with singularities on G .

2.0.1. *The function $\phi_s^{(2)}$.* There exists a unique family of functions $\phi_s^{(2)}$ ($\text{Re}(s) > n/2$) such that

- $\phi_s^{(2)}$ is a C^∞ -function on $G - HK$ and (H, K) -invariant, i.e.,

$$\phi_s^{(2)}(h g k) = \phi_s^{(2)}(g) \quad \forall h \in H, \forall g \in G - HK, \forall k \in K.$$

- $\phi_s^{(2)}$ satisfies the differential equation

$$\Omega \phi_s^{(2)}(g) = (s^2 - (n/2)^2) \phi_s^{(2)}(g), \quad g \in G - HK.$$

- There exists a positive δ such that $\phi_s^{(2)}(\exp(tY_0)) - \log(t)$ is bounded on the interval $(0, \delta)$.
- $\phi_s^{(2)}(a_t)$ decays exponentially as t getting large:

$$\phi_s^{(2)}(a_t) = O(e^{-(\text{Re}(s) + n/2)t}) \quad (t \rightarrow +\infty).$$

([1, Proposition 2.4.2]).

We have the explicit formula:

$$\begin{aligned} \phi_s^{(2)}(a_t) = & -\frac{1}{2} \frac{\Gamma((s + n/2)/2) \Gamma((s - n/2)/2 + 1)}{\Gamma(s + 1)} \\ & \times (\cosh t)^{-(s+n/2)} {}_2F_1\left(\frac{s + n/2}{2}, \frac{s - n/2}{2} + 1; s + 1; \frac{1}{\cosh^2}\right), \quad (t > 0). \end{aligned}$$

([1, 2.5]).

2.0.2. *The function ψ_s .* Let \mathfrak{p}_\pm be the $\pm\sqrt{-1}$ -eigen space of the complex linear extension of J to $\mathfrak{p}_\mathbb{C}$. Then $\mathfrak{p}_+ = \sum_{i=0}^{n-1} \mathbb{C} X_i$ and $\mathfrak{p}_- = \sum_{i=0}^{n-1} \mathbb{C} \bar{X}_i$ with

$$X_0 = E_{n,n+1} + E_{n+1,n} + \sqrt{-1}(E_{n,n+2} + E_{n+2,n}),$$

$$X_i = E_{i,n+1} + E_{n+1,i} + \sqrt{-1}(E_{i,n+2} + E_{n+2,i}), \quad 1 \leq i \leq n-1.$$

Let $\{\omega_i\}$ and $\{\bar{\omega}_i\}$ be the dual basis of $\{X_i\}$ and $\{\bar{X}_i\}$ respectively. Put

$$\mathbf{v}_{11} := \frac{1}{4} \left(\sum_{i=1}^{n-1} \omega_i \wedge \bar{\omega}_i - (n-1)\omega_0 \wedge \bar{\omega}_0 \right) (\in \mathfrak{p}_+^* \wedge \mathfrak{p}_-^*)$$

Then $(\mathfrak{p}_+^* \wedge \mathfrak{p}_-^*)^M$ is a two dimensional space generated by \mathbf{v}_{11} and the Kähler form $\bar{\omega} = \frac{\sqrt{-1}}{2} \sum_{i=0}^{n-1} \omega_i \wedge \bar{\omega}_i$. For $\text{Re}(s) > n/2$, put

$$\psi_s(g) = \frac{1}{4} \sum_{i,j=0}^{n-1} R_{X_i \bar{X}_j} \phi_s^{(2)}(g) \omega_i \wedge \bar{\omega}_j \quad g \in G - HK.$$

Here are some properties of the function ψ_s .

- ψ_s is a C^∞ -function on $G - HK$ such that

$$\psi_s(hgk) = (\text{Ad}_{\mathfrak{p}_+}^* \wedge \text{Ad}_{\mathfrak{p}_-}^*)(k)^{-1} \psi_s(g), \quad \forall h \in H, \forall g \in G - HK, \forall k \in K.$$

Here $\text{Ad}_{\mathfrak{p}_\pm}^*$ be the coadjoint representation of K on \mathfrak{p}_\pm^* .

- We have $\psi_s(a_t) = f_s(t) \mathbf{v}_{11}$ with

$$f_s(t) = \left(\tanh t \frac{d}{dt} - \frac{s^2 - (n/2)^2}{n} \right) \phi_s^{(2)}(a_t), \quad t > 0.$$

- There exists a positive δ such that $f_s(t) + \frac{s^2 - (n/2)^2}{2n} \log t$ is bounded on the interval $(0, \delta)$.
- We have the estimation:

$$f_s(t) \prec e^{-(\text{Re}(s) + n/2)t}, \quad t \in [1, \infty).$$

3. CURRENTS DEFINED BY POINCARÉ SERIES

Let Γ be as in the introduction. For $\alpha \in A(\Gamma \backslash G/K)$, we have a unique C^∞ -function $\tilde{\alpha} : G \rightarrow \bigwedge \mathfrak{p}_\mathbb{C}^*$ such that $\tilde{\alpha}(\gamma g k) = \tau(k)^{-1} \tilde{\alpha}(g)$, ($\gamma \in \Gamma, k \in K$) and such that

$$(1) \quad \langle (\pi^* \alpha)(gK), dL_g(\xi_o) \rangle = \langle \tilde{\alpha}(g), \xi_o \rangle, \quad g \in G, \xi_o \in \bigwedge \mathfrak{p} = \bigwedge T_o(G/K)$$

holds. Here L_g denotes the left translation on G/K by the element g and we identify \mathfrak{p} with $T_o(G/K)$, the tangent space of G/K at $o = eK$. Let dk (resp. dk_0) be the normalized Haar measure of K (resp. K_H) with total volume 1. Then there exists a Haar measure dg (resp. dh) of G (resp. H) such that $\frac{dg}{dk}$ (resp. $\frac{dh}{dk_0}$) corresponds to the measure of the symmetric space G/K (resp. H/K_H) determined by the invariant volume form associated to the Kähler form.

For any left Γ -invariant function f on G , put

$$\mathcal{I}_H(f; g) = \int_{\Gamma_H \backslash H} f(hg) dh, \quad g \in G.$$

Let $\varphi_s = \phi_s^{(2)}$ ($\text{Re}(s) > n/2$) or ψ_s ($\text{Re}(s) > n/2$). Then the integral

$$\int_{\Gamma \backslash G} \left(\sum_{\gamma \in \Gamma_H \backslash \Gamma} \|\varphi_s(\gamma g)\| \right) dg$$

is locally bounded in $\text{Re}(s) > n/2$ ([1, Proposition 3.1.1]), and there exists a unique current $P(\varphi_s)$ on $\Gamma \backslash G/K$ such that

$$\begin{aligned} \langle P(\varphi_s), * \tilde{\alpha} \rangle &= \int_{\Gamma \backslash G} \left(\sum_{\gamma \in \Gamma_H \backslash \Gamma} \varphi_s(\gamma g) | \tilde{\alpha}(g) \right) dg \\ &= \frac{\pi}{2} \int_0^\infty (\varphi_s(a_t) | \mathcal{I}_H(\tilde{\alpha}; a_t)) \sinh t (\cosh t)^{n-1} dt, \quad \forall \alpha \in A_c(\Gamma \backslash G/K) \end{aligned}$$

Here $(\cdot | \cdot)$ is the Hermitian inner product of $\mathfrak{p}_\mathbb{C}^*$ canonically induced by the inner product $(8n)^{-1} B(X, Y)$ on \mathfrak{p} .

We have the current $G_s := P(\phi_s^{(2)})$ of $(0, 0)$ -type and the one $\Psi_s := P(\psi_s)$ of $(1, 1)$ -type on $\Gamma \backslash G/K$ which depends holomorphically on $\text{Re}(s) > n/2$.

4. DIFFERENTIAL EQUATIONS

Let $\operatorname{Re}(s) > n/2$. Then the currents G_s and Ψ_s satisfy the differential equations:

$$\begin{aligned}\Delta G_s &= -((2s)^2 - n^2) G_s - 2\pi \Lambda \delta_D, \\ \Delta \Psi_s &= -((2s)^2 - n^2) \left(\Psi_s - \frac{\pi\sqrt{-1}}{4} \delta_D - \frac{\pi\sqrt{-1}}{4n} L\Lambda \delta_D \right), \\ \partial\bar{\partial}G_s + \pi\sqrt{-1}\delta_D &= \frac{\sqrt{-1}}{2n} ((2s)^2 - n^2) L G_s + 4\Psi_s.\end{aligned}$$

Here Λ is the adjoint of the Lefschets operator $L\alpha = \omega_{\Gamma\backslash G/K} \wedge \alpha$ ([1, Theorem 7.6.1]).

5. MEROMORPHICITY

Suppose $\Gamma\backslash G$ is compact. Let $\{\lambda_m\}_{m \in \mathbb{N}}$ be the increasing sequence of the eigenvalues of the negative of the Casimir operator $-R_\Omega$ acting on $L^2(\Gamma\backslash G/K)$ such that each eigenvalue occurs with its multiplicity. We fix an orthonormal basis $\{\varphi_m\}_{m \in \mathbb{N}}$ of $L^2(\Gamma\backslash G/K)$ consisting of automorphic forms on $\Gamma\backslash G/K$ such that $-R_\Omega \varphi_m = \lambda_m \varphi_m$ ($\forall m \in \mathbb{N}$). Then we have the spectral expansion of G_s ($\operatorname{Re}(s) > n/2$):

$$(2) \quad \langle G_s, *\bar{\alpha} \rangle = \sum_{m=0}^{\infty} \frac{\mathcal{I}_H(\bar{\varphi}_m; e)}{(n/2)^2 - \lambda_m - s^2} \langle \varphi_m | \bar{\alpha} \rangle_{L^2}, \quad \alpha \in A_c(\Gamma\backslash G/K).$$

Here $\langle \cdot | \cdot \rangle_{L^2}$ is the L^2 -inner product of $L^2(\Gamma\backslash G/K)$. The corresponding result for the ‘unitary case’ is proved in [1, Proposition 6.2.2]. The proof for the present case is parallel since we assume $\Gamma\backslash G$ is compact. Then by an estimation similar to that in [1, Theorem 6.2.1 (1)], the series (2) is absolutely convergent for an arbitrary $s \in \{s \in \mathbb{C} \mid s^2 \neq (n/2)^2 - \lambda_m (\forall m)\}$ locally uniformly. Hence the current $s \mapsto G_s$, which is originally holomorphic only on $\operatorname{Re}(s) > n/2$, has a meromorphic continuation to the whole s -plane with possible simple poles at the points $s \in \mathbb{C}$ such that $s^2 = (n/2)^2 - \lambda_m$ ($\exists m$).

6. GREEN CURRENT

The point $s = n/2$ is a simple pole of G_s with the residue

$$\operatorname{Res}_{s=n/2} G_s = -\frac{1}{n} \frac{\operatorname{vol}(\Gamma_H \backslash H)}{\operatorname{vol}(\Gamma \backslash G)},$$

a constant function on $\Gamma\backslash G/K$.

Definition

We put \mathcal{G} to be $(-2\pi)^{-1}$ times the constant term of the Laurent expansion of G_s at $s = n/2$, i.e.,

$$\mathcal{G}(x) = \frac{-1}{2\pi} \lim_{s \rightarrow n/2} \left(G_s(x) - \frac{\kappa}{s - n/2} \right)$$

with $\kappa = -\frac{1}{n} \frac{\operatorname{vol}(\Gamma_H \backslash H)}{\operatorname{vol}(\Gamma \backslash G)}$.

Theorem

- The current-valued function $s \mapsto \Psi_s$ on $\operatorname{Re}(s) > n/2$ has a meromorphic continuation to the whole s -plane. The point $s = n/2$ is a regular point of the meromorphic function Ψ_s and the value $\Psi_{n/2}$ is harmonic, i.e.,

$$\Delta \Psi_{n/2} = 0.$$

- The current \mathcal{G} satisfies Green's equation:

$$\mathrm{dd}^c \mathcal{G} + \delta_D = \frac{1}{\pi} (\kappa \omega_{\Gamma \backslash G/K} + 4\Psi_{n/2}).$$

REFERENCES

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